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Technical Report

Modeling and Worst-Case Dimensioning of Cluster-Tree Wireless Sensor Networks: Proofs and computation details

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TR-060601 (Annex for RTSS 2006: proofs and computation details)

Version: 1.0

Date: 01 June 2006

GTS Allocation Analysis in IEEE 802.15.4 for Real-Time Wireless Sensor Networks

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Abstract

This document is associated to the paper entitled "Modeling and Worst-Case Dimensioning of Cluster-Tree Wireless Sensor Networks" submitted to RTSS 2006. Although not fundamental for the self-containment of the paper, it provides the proofs of the corollaries used in the paper, and some computational details that may be helpful for the reviewing process. The text with a gray background presents the details not included in the paper, namely the proofs and the computation details.

Modeling and Worst-Case Dimensioning of Cluster-Tree Wireless Sensor Networks: proofs and computation details

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Abstract

This document is associated to the paper entitled "Modeling and Worst-Case Dimensioning of Cluster-Tree Wireless Sensor Networks" submitted to RTSS 2006. Although not fundamental for the self-containment of the paper, it provides the proofs of the corollaries used in the paper, and some computational details that may be helpful for the reviewing process. The text with a gray background presents the details not included in the paper, namely the proofs and the computation details.

1. Proof of Corollary 1

Corollary 1. Assume that a flow is constrained by an arrival curve $\alpha(t) = b + r \cdot t$ and a FIFO node provides a guaranteed service curve $\beta_{R,T}(t) = R \cdot (t - T)^+$ to the flow. Then, the output bound of the flow is expressed as:

$$\alpha^*(t) = \alpha(t) + r \cdot T \quad \textbf{(6)}$$

And for any constant $K \in \mathbb{R}$, we easily show that:

$$\left(K \cdot \alpha(t)\right) \odot \beta_{R,T}(t) = K.\left(\alpha \odot \beta_{R,T}\right)(t) \quad (7)$$

Proof of Eq. (6)
By definition, we have:

$$\alpha^*(t) = (\alpha \odot \beta)(t)$$

$$\alpha^*(t) = (\alpha \odot \beta)(t) = \sup_{s \geq 0}\left(\alpha(t+s) - \beta(s)\right)$$

Using the definition of α and β, we get:

$$\alpha^*(t) = \sup_{s \geq 0}\left(b + r \cdot (t+s) - R \cdot (s-T)^+\right)$$

$$\alpha^*(t) = \max\begin{pmatrix} \sup_{0 \leq s \leq T}\left(b + r \cdot (t+s) - R \cdot (s-T)^+\right), \\ \sup_{T \leq s}\left(b + r \cdot (t+s) - R \cdot (s-T)^+\right) \end{pmatrix}$$

$$\alpha^*(t) = \max\begin{pmatrix} b + r \cdot (t+T) - R \cdot (T-T)^+, \\ b + r \cdot (t+T) - R \cdot (T-T)^+ \end{pmatrix}$$

$$\alpha^*(t) = b + r \cdot (t+T) = \alpha(t) + r \cdot T$$

□

Proof of Eq. (7)

$$\left(K \cdot \alpha(t)\right) \odot \beta_{R,T}(t) = \left(K \cdot (b + r \cdot t)\right) \odot \beta_{R,T}(t)$$

$$\left(K \cdot \alpha(t)\right) \odot \beta_{R,T}(t) = \left(K \cdot b + K \cdot r \cdot t\right) \odot \beta_{R,T}(t)$$

Let us denote $b_K = K \cdot b$ and $r_K = K \cdot r$

$$\left(K \cdot \alpha(t)\right) \odot \beta_{R,T}(t) = \left(b_K + r_K \cdot t\right) \odot \beta_{R,T}(t)$$

According to Eq. (6), we have:

$$\begin{aligned}\left(b_K + r_K \cdot t\right) \odot \beta_{R,T}(t) &= b_K + r_K \cdot t + r_K \cdot T \\ &= K \cdot b + K \cdot r \cdot t + K \cdot r \cdot T \\ &= K \cdot \left(b + r \cdot t + r \cdot T\right) \\ &= K \cdot \left(\alpha(T) \odot \beta_{R,T}(t)\right)\end{aligned}$$

□

2. Computation of Input and Output Flows

Analysis of depth *maxDepth+1* (*depth* = 4)

At depth *maxDepth*+1, there is no router, and there are nodes with input data flows, each flow constrained by the arrival curve $\alpha_{data}(t)$. Since each node is granted a service curve $\beta_{data}(t)$, then using Eqs. (5) and (6), the output flow of each child node can be expressed as follows:

$$\alpha^*_{data}(t) = \left(\alpha_{data} \odot \beta_{data}\right)(t) = \alpha_{data}(t) + r_{data} \cdot T_{data} \quad \textbf{(1)}$$

Analysis of depth m*axDepth* (*depth* = 3)

At depth *maxDepth*, the total input of each router, denoted by $\overline{\alpha}_{maxDepth}(t)$, comprises its sensory data flow constrained by $\alpha_{data}(t)$, and the sum of the output flows of its child nodes.

$$\overline{\alpha}_{maxDepth}(t) = \alpha_{data}(t) + N_{Child} \cdot \alpha^*_{data}(t)$$

Thus, according to Eq. (12), we have:

$$\overline{\alpha}_{maxDepth}(t) = \left(N_{Child} + 1\right) \cdot \alpha_{data}(t) + N_{Child} \cdot r_{data} \cdot T_{data} \quad \textbf{(13)}$$

Note that $\overline{r}_{maxDepth} = \left(N_{child} + 1\right) \cdot r_{data}$ is the resulting rate of the aggregate of $\left(N_{child} + 1\right)$ input data flows, and

$\bar{b}_{maxDepth} = (N_{child}+1)\cdot b_{data} + N_{Child}\cdot r_{data}\cdot T_{data}$ is its resulting burst.

The input flow $\bar{\alpha}_{maxDepth}(t)$ is forwarded by the router at depth *maxDepth* to its parent router at depth *maxDepth*-1. This child router is allocated a service curve $\beta_{maxDepth-1}(t) = R_{maxDepth-1}\cdot(t - T_{maxDepth-1})^{+}$ by its parent. Hence, according to Eq. (5), the output flow from a child router at depth *maxDepth* is then expressed as:

$$\alpha^{*}_{maxDepth}(t) = \left(\bar{\alpha}_{maxDepth}(t) \odot \beta_{maxDepth-1}(t)\right)$$

As a result, applying Eq. (6) we get:

$$\alpha^{*}_{maxDepth}(t) = \left(\bar{\alpha}_{maxDepth}(t) + \sigma_{maxDepth-1}\right) \quad \textbf{(14)}$$

where $\sigma_{maxDepth-1} = \bar{r}_{maxDepth}\cdot T_{maxDepth-1}$

Analysis of depth *maxDepth*-1 (*depth* = 2)

At depth *maxDepth*-1, the total input of each router, denoted by $\bar{\alpha}_{maxDepth-1}(t)$, comprises its sensory data flow constrained by $\alpha_{data}(t)$, and the sum of the output flows of its child routers $\alpha^{*}_{maxDepth}(t)$ and the output of its child nodes $\alpha^{*}_{data}(t)$. It results that:

$$\bar{\alpha}_{maxDepth-1}(t) = \left(\left(\alpha_{data}(t) + N_{child}\cdot\alpha^{*}_{data}(t)\right) + N_{router}\cdot\alpha^{*}_{maxDepth}(t)\right)$$

$$\bar{\alpha}_{maxDepth-1}(t) = \left(\bar{\alpha}_{maxDepth}(t) + N_{router}\cdot\left(\bar{\alpha}_{maxDepth}(t) + \sigma_{maxDepth-1}\right)\right)$$

Thus, according to Eqs. (13) and (14) we have:

$$\bar{\alpha}_{maxDepth-1}(t) = \left((N_{router}+1)\cdot\bar{\alpha}_{maxDepth}(t) + N_{router}\cdot\sigma_{maxDepth-1}\right) \quad \textbf{(15)}$$

The input flow $\bar{\alpha}_{maxDepth-1}(t)$ is forwarded by the router at depth *maxDepth*-1 to its parent router at depth *maxDepth*-2. This child router is allocated a service curve $\beta_{maxDepth-2}(t) = R_{maxDepth-2}\cdot(t - T_{maxDepth-2})^{+}$ by its parent. Hence, according to Eq. (5), the output flow from a child router at depth *maxDepth*-1 is then expressed as:

$$\alpha^{*}_{maxDepth-1}(t) = \bar{\alpha}_{maxDepth-1}(t) \odot \beta_{maxDepth-2}(t)$$

As a result, applying Eq (15):

$$\alpha^{*}_{maxDepth-1}(t) = \begin{pmatrix}(N_{router}+1)\cdot\bar{r}_{maxDepth}\cdot t \\ +(N_{router}+1)\cdot\bar{b}_{maxDepth} + N_{router}\cdot\sigma_{maxDepth-1}\end{pmatrix} \odot \beta_{maxDepth-2}(t)$$

Applying Eq. (6), we have:

$$\alpha^{*}_{maxDepth-1}(t) = \begin{pmatrix}(N_{router}+1)\cdot\bar{r}_{maxDepth}\cdot t + (N_{router}+1)\cdot\bar{b}_{maxDepth} \\ +N_{router}\cdot\sigma_{maxDepth-1} + (N_{router}+1)\cdot\bar{r}_{maxDepth}\cdot T_{maxDepth-2}(t)\end{pmatrix}$$

Thus, we get:

$$\alpha^{*}_{maxDepth-1}(t) = \begin{pmatrix}(N_{router}+1)\cdot\bar{\alpha}_{maxDepth}(t) + \\ N_{router}\cdot\sigma_{maxDepth-1} + \sigma_{maxDepth-2}\end{pmatrix} \quad \textbf{(16)}$$

where $\sigma_{maxDepth-2} = (N_{router}+1)\cdot\bar{r}_{maxDepth}\cdot T_{maxDepth-2}$

Analysis of depth *maxDepth*-1 (*depth* = 1)

At depth *maxDepth*-2, the total input of each router, denoted by $\bar{\alpha}_{maxDepth-2}(t)$, comprises its sensory data flow constrained by $\alpha_{data}(t)$, and the sum of the output flows of its child routers $\alpha^{*}_{maxDepth-1}(t)$ and the output of its child nodes $\alpha^{*}_{data}(t)$. It results that:

$$\bar{\alpha}_{maxDepth-2}(t) = \left(\left(\alpha_{data}(t) + N_{child}\cdot\alpha^{*}_{data}(t)\right) + N_{router}\cdot\alpha^{*}_{maxDepth-1}(t)\right)$$

Hence, using Eq. (16), we get

$$\bar{\alpha}_{maxDepth-2}(t) = \left(\bar{\alpha}_{maxDepth}(t) + N_{router}\cdot\begin{pmatrix}(N_{router}+1)\cdot\bar{\alpha}_{maxDepth}(t) \\ +N_{router}\cdot\sigma_{maxDepth-1} \\ +\sigma_{maxDepth-2}\end{pmatrix}\right)$$

It results that:

$$\bar{\alpha}_{maxDepth-2}(t) = \begin{pmatrix}(N^{2}_{router} + N_{router} + 1)\cdot\bar{\alpha}_{maxDepth}(t) \\ N^{2}_{router}\cdot\sigma_{maxDepth-1} + N_{router}\cdot\sigma_{maxDepth-2}\end{pmatrix} \quad \textbf{(17)}$$

and the output flow from a child router at depth *maxDepth*-2 for a service curve $\beta_{maxDepth-3}(t)$ is then expressed as:

$$\alpha^{*}_{maxDepth-2}(t) = \bar{\alpha}_{maxDepth-2}(t) \odot \beta_{maxDepth-3}(t)$$

As a result, applying Eq (17):

$$\alpha^{*}_{maxDepth-2}(t) = \begin{pmatrix}(N^{2}_{router} + N_{router} + 1)\cdot\bar{\alpha}_{maxDepth}(t) + \\ N^{2}_{router}\cdot\sigma_{maxDepth-1} + N_{router}\cdot\sigma_{maxDepth-2}\end{pmatrix} \odot \beta_{maxDepth-3}(t)$$

Applying Eq. (6), we have:

$$\alpha^{*}_{maxDepth-2}(t) = \begin{pmatrix}(N^{2}_{router} + N_{router} + 1)\cdot\bar{\alpha}_{maxDepth}(t) + \\ N^{2}_{router}\cdot\sigma_{maxDepth-1} + N_{router}\cdot\sigma_{maxDepth-2} \\ +(N^{2}_{router} + N_{router} + 1)\cdot\bar{r}_{maxDepth}\cdot\sigma_{maxDepth-3}\end{pmatrix}$$

Thus, we obtain:

$$\alpha^{*}_{maxDepth-2}(t) = \left(\bar{\alpha}_{maxDepth-2}(t) + \sigma_{maxDepth-3}\right) \text{ where} \quad \textbf{(18)}$$

$$\sigma_{maxDepth-3} = (N^{2}_{router} + N_{router} + 1)\cdot\bar{r}_{maxDepth}\cdot T_{maxDepth-3}$$

General expressions of input/output flows for depth *maxDepth*-i

By recurrence, we can easily prove that the input flow of each router at depth (*maxDepth-i*) is expressed as follows:

$$\overline{\alpha}_{maxDepth-i}(t) = \left(\sum_{j=0}^{i} N_{router}^{j}\right)\cdot\overline{\alpha}_{maxDepth}(t) + \sum_{j=0}^{i-1}\left(N_{router}^{i-j}\cdot\sigma_{maxDepth-(j+1)}\right) \quad \textbf{(19)}$$

$$\text{where } \sigma_{maxDepth-n} = \left(\sum_{k=0}^{n-1} N_{router}^{k}\right)\cdot\overline{r}_{maxDepth}\cdot T_{maxDepth-n}$$

and the output flow from a child router at depth (*maxDepth-i*) for a service curve $\beta_{maxDepth-(i+1)}(t)$ is then expressed as:

$$\alpha^{*}_{maxDepth-i}(t) = \overline{\alpha}_{maxDepth-i}(t) + \sigma_{maxDepth\text{-}(i+1)} = \left(\left(\sum_{j=0}^{i} N_{router}^{j}\right)\cdot\overline{\alpha}_{maxDepth}(t) + \sum_{j=0}^{i}\left(N_{router}^{i-j}\cdot\sigma_{maxDepth\text{-}(j+1)}\right)\right) \quad \textbf{(2)}$$

3. Delay Bound Analysis

***The Second Approach (tighter delay bounds*)**

The idea of the second approach is to use the aggregate scheduling corollary based on Eq. (11) and the service curve concatenation theorem based on Eq. (8). First, we aim to derive the service curve offered to a particular individual flow *F* among the aggregate by a router at a given depth, using Eq. (11). Then, we deduce the *equivalent service curve* for this particular flow along the path, using Eq. (8). The delay bound will be computed based on the equivalent service curve.

We consider the tandem of service curve elements as presented in Fig. 5. The approach is based on the following algorithm:

- *Step 1.* β_{last} is equal to the last service curve element (i.e. router) in the tandem.
- *Step 2.* Compute the β_{eq} *equivalent service curve* to an output flow of the previous service curve element $\beta_{\text{last-1}}$ using Eq. (11).
- *Step 3.*Replace $\beta_{\text{last}} = \beta_{\text{last-1}}\otimes\beta_{\text{eq}}$ since the concatenation is also a service curve to the input of $\beta_{\text{last-1}}$. The length of the tandem is then reduced by one.
- *Step 4.* if the tandem length is greater than one, then Go to *Step 1*; else, β_{last} is the equivalent end-to-end service curve.
- *Step 5.* Compute the delay bound using the equivalent service curve applied to the input arrival curve.

Computation of the delay bound using the end-to-end service curve

We propose to apply this methodology in the case of our cluster-tree WSN model. Consider the example in Fig. 1.

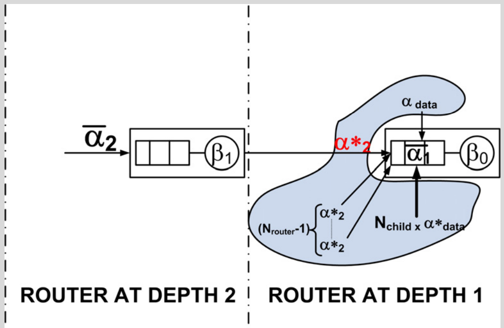

Fig. 1. Aggregate flows at the input of a service curve element

To illustrate how to compute the recurrent equation, we consider the system in Fig. 1, showing the input, output and service curve at a router at a depth 1, one hop before the root. $\overline{\alpha}_1$ is composed of N_{router} instances of α_2^*, N_{child} instances of α_{data}^*, and one instance of the input data flow α_{data}. The first objective is to derive a service curve for one flow α_2^* using the aggregation scheduling theorem. After doing so, we can deduce the overall service curve offered to $\overline{\alpha}_2$ based on the concatenation theorem.

We denote by $\omega(maxDepth - j)$ the tandem of parent-child routers **from depth (*maxDepth-j*) to the depth = 1**. Hence, $\omega(1)$ denotes the router at depth 1, $\omega(2)$ denote the link from router at depth 2 its parent router at depth 1, etc.

Computing the Service Curve for α_2^* at depth 1 (*j* = *maxDepth*-1)

Let us consider $\Phi_1(t) = \overline{\alpha}_1(t) - \alpha_2^*(t)$

Hence, $\beta_0(t)$ is a service curve offered to the aggregate of flows $\Phi_1(t) = \overline{\alpha}_1(t) - \alpha_2^*(t)$ and α_2^*. Note that $\Phi_1(t) = \overline{\alpha}_1(t) - \alpha_2^*(t)$ is a linear arrival curve. Hence, the aggregation scheduling theorem can be applied in this case, and it results that, (here we consider $\theta = 0$, in Eq. (11)).

$$\beta_{\omega(1)}^{eq,\alpha_2^*}(t) = \left(R_0 - \left(\overline{r}_1 - r_2^*\right)\right) \cdot \left[t - \left(\frac{\left(\overline{b}_1 - b_2^*\right) + R_0 \cdot T_0}{R_0 - \left(\overline{r}_1 - r_2^*\right)} \right) \right]^+ \quad \textbf{Eq. (30)}$$

Now, $\beta_{\omega(1)}^{eq,\alpha_2^*}$ and β_1 are service curves offered to $\overline{\alpha}_2$, hence, their convolution describes the equivalent service curve offered by the tandem $\omega(2)$ to the input $\overline{\alpha}_2$. Thus, we obtain:

$$\beta_{\omega(2)}^{eq,\overline{\alpha}_2}(t) = \beta_1(t) \otimes \beta_{\omega(1)}^{eq,\alpha_2^*}(t) \quad \textbf{Eq. (31)}$$

According to [1] (Theorem 2.1.5, concatenation of rate-latency service curves), the concatenation of two rate-latency service curve β_1 and β_2, is a rate-latency service curve β where the equivalent rate $R = \min(R_1, R_2)$ and the equivalent latency is $T = T_1 + T_2$. Thus, Eq. (31) can be re-written as:

$$\beta_{\omega(2)}^{eq,\overline{\alpha}_2}(t) = R_{\omega(2)}^{eq,\overline{\alpha}_2}\left(t - T_{\omega(2)}^{eq,\overline{\alpha}_2}\right)^+ \text{ where} \quad \textbf{Eq. (32)}$$

$$R_{\omega(2)}^{eq,\overline{\alpha}_2} = \min\left(R_1, R_{\omega(1)}^{eq,\alpha_2^*}\right) \text{ and } T_{\omega(2)}^{eq,\overline{\alpha}_2} = T_1 + T_{\omega(1)}^{eq,\alpha_2^*}$$

Now, the system length is reduced by one. We consider the new system to derive the service curve for the flow α_3^*, by considering $\Phi_2(t) = \overline{\alpha}_2(t) - \alpha_3^*(t)$ which receives together with α_3^* the service curve $R_{\omega(2)}^{eq,\overline{\alpha}_2}$.

Let us consider $\Phi_2(t) = \overline{\alpha}_2(t) - \alpha_3^*(t)$

Hence, $\beta_{\omega(2)}^{eq,\overline{\alpha}_2}(t)$ is a service curve offered to the aggregate of flows $\Phi_2(t) = \overline{\alpha}_2(t) - \alpha_3^*(t)$ and $\alpha_3^*(t)$. Note that

$\Phi_2(t) = \overline{\alpha}_2(t) - \alpha_3^*(t)$ is a linear arrival curve. Hence, the aggregation scheduling theorem can be applied in this case, and it results that, (here we consider $\theta = 0$, in Eq. (11)).

$$\beta_{\omega(2)}^{eq,\alpha_3^*}(t) = \left(R_{\omega(2)}^{eq,\overline{\alpha}_2} - \left(\overline{r}_2 - r_3^*\right)\right) \cdot \left[t - \left(\frac{\left(\overline{b}_2 - b_3^*\right) + R_{\omega(2)}^{eq,\overline{\alpha}_2} \cdot T_{\omega(2)}^{eq,\overline{\alpha}_2}}{R_{\omega(2)}^{eq,\overline{\alpha}_2} - \left(\overline{r}_2 - r_3^*\right)} \right) \right]^+ \quad \textbf{Eq. (33)}$$

Now, $\beta_{\omega(2)}^{eq,\alpha_3^*}$ and β_2 are service curves offered to $\overline{\alpha}_3$, hence, their convolution describes the equivalent service curve offered by the tandem $\omega(3)$ to the input $\overline{\alpha}_3$. Thus, we obtain:

$$\beta_{\omega(3)}^{eq,\overline{\alpha}_3}(t) = \beta_2(t) \otimes \beta_{\omega(2)}^{eq,\alpha_3^*}(t) \quad \textbf{Eq. (34)}$$

We propose the following recursive algorithm to compute the end to end service curve for the cluster-tree topology:

```
initialization: βeq (t) = β0(t);
for i = 0 to maxDepth-2
    βlast (t) = βeq (t);
    Φ(t) = ᾱ(i+1)(t) - α*(i+2)(t)
    βeqAGGR (t) = Compute the resulting service curve for α*(i+2)(t) using aggregate
    scheduling theorem applied to βlast (t) and Φ(t).
     βeq (t) = concat(β(i+1) (t), βeqAGGR (t))
endfor
βeq (t) = concat(βmaxDepth (t), βeq (t))
Compute the delay bound using Eq. (3) applied to βeq (t) and ᾱmaxDepth .
```

Using this simple recursive algorithm, it is possible to compute the delay bound of the input flow of the router at the lowest depth. This delay bound is the maximum bound for the entire network.

We can deduce the following recurrent equations. At a given depth (*maxDepth-i*), we have: for $i = (maxDepth - 1)$ *downto* 0:

$$\Phi_{maxDepth-i}(t) = \overline{\alpha}_{maxDepth-i}(t) - \alpha^*_{maxDepth-(i-1)}(t)$$

$$\beta_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}}(t) = R_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}} \cdot \left(t - T_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}}\right)^+ \quad \text{where}$$

$$R_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}} = R_{\omega(\max Depth-i)}^{eq,\overline{\alpha}_{\text{maxDepth}-i}}(t) - \left(\overline{r}_{\text{maxDepth}-i} - r^*_{\text{maxDepth}-(i-1)}\right) \quad \text{and}$$

$$T_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}} = \frac{\left(\overline{b}(\max Depth-i) - b^*_{\text{maxDepth}-(i-1)}\right) + R_{\omega((\max Depth-i))}^{eq,\overline{\alpha}_{(\max Depth-i)}} \cdot T_{\omega((\max Depth-i))}^{eq,\overline{\alpha}_{(\max Depth-i)}}}{R_{\omega(\max Depth-i)}^{eq,\overline{\alpha}_{(\max Depth-i)}} - \left(\overline{r}(\max Depth-i) - r^*_{\text{maxDepth}-(i-1)}\right)}$$

and then, the equivalent service curve is computed as:

$$\beta_{\omega(\max Depth-(i-1))}^{eq,\overline{\alpha}_{\text{maxDepth}-(i-1)}}(t) = \beta_{\max Depth-i}(t) \otimes \beta_{\omega(\max Depth-i)}^{eq,\alpha^*_{\text{maxDepth}-(i-1)}}(t)$$

References

[1] J.-Y. Leboudec and P. Thiran, *A Theory of Deterministic Queuing Systems for the Internet*: Lecture Notes in Computer Science (LNCS), Vol. 2050, 2001.